

Inclusions of second quantization algebras

Franca Figliolini and Daniele Guido

1. Introduction.

In this note we study inclusions of second quantization algebras, namely inclusions $\mathcal{R}(M_0) \subset \mathcal{R}(M_1)$ of von Neumann algebras on the Fock space $e^{\mathcal{H}}$ (\mathcal{H} is a separable complex Hilbert space) generated by the Weyl unitaries with test functions in the closed, real linear subspaces M_0, M_1 of \mathcal{H} . More precisely we concentrate our attention to the case where both $\mathcal{R}(M_0)$ and $\mathcal{R}(M_1)$ are standard w.r.t. the vacuum vector $e^0 \in e^{\mathcal{H}}$, since in this case the tower and tunnel associated with the inclusion (and the corresponding relative commutants) can themselves be realized as second quantization algebras on the same space:

$$\cdots \subset \mathcal{R}(M_{-1}) \subset \mathcal{R}(M_0) \subset \mathcal{R}(M_1) \subset \mathcal{R}(M_2) \cdots$$

First we show that the class of irreducible inclusions of standard second quantization algebras is non empty, and that they are depth two inclusions, namely $\mathcal{R}(M_0)' \cap \mathcal{R}(M_3)$ is a factor. Then we prove that, when $M_0 \subset M_1$ is a (not necessarily irreducible) inclusion of standard spaces with finite codimension n , $\mathcal{R}(M)$ is isomorphic to the cross product of $\mathcal{R}(N)$ with \mathbb{R}^n . On the contrary, when the codimension is infinite, we show that the inclusion may be non regular (see subsection 4.1).

Second quantization algebras and their inclusions occur when studying algebras of local observables for the free fields. Inclusions of local observable algebras are in general neither irreducible nor come from a finite codimension inclusion of real vector spaces. In [9] however, local algebras for conformal field theories on the real line are studied, and it is shown that the inclusion of the real vector space corresponding to a bounded interval for the $n + p$ -th derivative of the current algebra into the real vector space for the same interval and the n -th derivative theory has codimension p (and is irreducible when $p = 1$). We show in Theorem 4.1 that the corresponding inclusion of second quantization algebras is given by a cross product for any $n \geq 0, p > 0$.

Our analysis was also motivated by results concerning depth two inclusions of von Neumann algebras. It is well known that, analyzing the Jones' tower associated with the inclusion of a von Neumann algebra \mathcal{N} into the cross product \mathcal{M} of the same algebra with an outer action of locally compact group, the family of

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relative commutants presents some characteristic features, namely the first relative commutant is trivial, the second is abelian, and the third is a type I factor. The study of inclusions with these properties, started in [10], was recently enriched by a pair of papers [5, 6] where these properties, together with the existence of an operator valued weight from \mathcal{M}_1 to \mathcal{M}_0 , were shown to characterize the inclusions given by cross products with locally compact groups. We show that inclusions of second quantization algebras may produce examples of irreducible depth two inclusions with type III third relative commutant. By a result in [5], these inclusions are not regular, since in that case the third relative commutant would be type I, hence do not correspond to a crossed-product with a locally compact group.

On the other hand, our examples satisfy many of the features of a crossed-product, the locally compact group being replaced by an infinite-dimensional vector space, thus furnishing examples in order to develop a theory of non-locally compact cross products.

2. Preliminaries

In this section \mathcal{H} will be a separable Hilbert space and $e^{\mathcal{H}}$ the *symmetric Fock space* over it, i.e. the space

$$e^{\mathcal{H}} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$$

where $\mathcal{H}^{\otimes n}$ is the subspace of the n -th tensor product of \mathcal{H} which is pointwise invariant under the natural action of the permutation group.

The set of *coherent* vectors in $e^{\mathcal{H}}$ consists of the vectors

$$e^h = \bigoplus_{n=0}^{\infty} \frac{h^{\otimes n}}{\sqrt{n!}} \quad .$$

This set turns out to be total in $e^{\mathcal{H}}$ (see e.g. [8], p.32).

There are two important classes of operators acting on $e^{\mathcal{H}}$:

Second quantization operators

$$e^A = \bigoplus_{n=0}^{\infty} A^{\otimes n} \quad ,$$

where A is a densely defined, closed operator on \mathcal{H} , and *Weyl unitaries*, which are the range of the map

$$h \rightarrow W(h)$$

from \mathcal{H} to the unitaries on $e^{\mathcal{H}}$ defined by

$$(2.1) \quad W(h)e^0 = \exp\left(-\frac{1}{4}\|h\|^2\right) e^{\frac{i}{\sqrt{2}}h}, \quad h \in \mathcal{H}$$

$$(2.2) \quad W(h)W(k) = \exp\left(-\frac{i}{2}\text{Im}(h, k)\right) W(h+k) \quad h, k \in \mathcal{H}$$

The vector e^0 is called *vacuum* and the relations in the last equality are called Canonical Commutation Relations. Via the preceding equalities $W(h)$ becomes a well defined, isometric and invertible (with inverse $W(-h)$) operator on the dense set spanned by coherent vectors, and hence it extends to a unitary on $e^{\mathcal{H}}$. Weyl unitaries generate the so-called *second quantization algebras*. With each closed real

linear subspace $K \subset \mathcal{H}$ (in the following we shall write $K \leq_{\mathbb{R}} \mathcal{H}$), a von Neumann algebra $\mathcal{R}(K)$ is associated, defined by

$$\mathcal{R}(K) = \{W(h), \quad h \in K\}''.$$

In the following Proposition we recall the basic facts about these algebras [1, 4]:

PROPOSITION 2.1. *Let us consider the map $K \rightarrow \mathcal{R}(K)$ where $K \leq_{\mathbb{R}} \mathcal{H}$. Then*
(i) The mentioned map is an isomorphism of complemented nets where “ \wedge ” is the intersection, “ \vee ” is the generated real subspace, resp. von Neuman algebra, and the complementation “ $'$ ” is the polar space w.r.t. $\text{Im}(\cdot, \cdot)$, resp. the commutant.
(ii) The map preserve the standard property, namely the vacuum vector is standard, i.e. cyclic and separating, for $\mathcal{R}(K)$ if and only if K is standard, i.e. $K \cap iK = 0$ and $K + iK$ is dense. Moreover the Tomita operator S associated with a standard algebra $\mathcal{R}(K)$ is the second quantization of the Tomita operator s on K defined by

$$s : h + ik \rightarrow h - ik \quad h, k \in K$$

and also $J = e^j$, $\Delta = e^\delta$, where $S = J\Delta^{1/2}$, $s = j\delta^{1/2}$.

As a consequence of the preceding proposition, the properties of the second quantization algebras can be studied in terms of the generating one-particle subspaces, and in the following we shall attribute to a real linear subspace K of \mathcal{H} all the properties of its second quantization algebra, for instance we will say that K is a standard type III subspace if $\mathcal{R}(K)$ is, etc. We shall also say that $M' = \{x \in \mathcal{H} : \text{Im}(x, y) = 0 \ \forall x \in M\}$ is the commutant of M . In particular, if $M_0 \subset M_1$ is an inclusion of standard subspaces of \mathcal{H} , the tower and the tunnel [11, 3] generated by the inclusion $\mathcal{M}_0 \equiv \mathcal{R}(M_0) \subset \mathcal{M}_1 \equiv \mathcal{R}(M_1)$ can be spatially realized as

$$\begin{aligned} \mathcal{M}_{i+1} &= J_i \mathcal{M}'_{i-1} J_i & i \geq 1 \\ \mathcal{M}_{i-1} &= J_i \mathcal{M}'_{i+1} J_i & i \leq 0, \end{aligned}$$

where J_i is the Tomita conjugation for $\{\mathcal{M}_i, e^0\}$. Therefore they are the second quantization of the tower, resp. tunnel, of standard subspaces:

$$\begin{aligned} M_{i+1} &= j_i M'_{i-1} & i \geq 1 \\ M_{i-1} &= j_i M'_{i+1} & i \leq 0. \end{aligned}$$

3. Inclusions of standard subspaces

DEFINITION 3.1. We shall say that a pair (E, F) , $E, F \leq_{\mathbb{R}} \mathcal{H}$, is a standard pair if $E \wedge F = \{0\}$ and $E \vee F = \mathcal{H}$. A standard pair will be called strongly standard if, moreover, $E + F = \mathcal{H}$. If (E, F) is a standard pair, we define the operator $s_{E,F}$ as follows:

$$\begin{aligned} s_{E,F} : \quad E + F &\rightarrow E + F \\ e + f &\mapsto e - f \end{aligned}$$

Let us observe that N is standard if and only if (N, iN) is a standard pair. Also, N is a factor if and only if (N, N') is a standard pair. Indeed, if N is a factor then $N \cap N' = \{0\}$, and, taking the commutant, $N \vee N' = (N \cap N')' = \mathcal{H}$.

LEMMA 3.2. *Let (E, F) be a standard pair. Then:*

- (i) $s_{E,F}$ is closed
- (ii) $s_{E,F}$ is bounded iff (E, F) is a strongly standard pair.

(iii) The pair (E, F) can be properly extended by a standard pair (\tilde{E}, \tilde{F}) (i.e. $E \subseteq \tilde{E}$, $F \subseteq \tilde{F}$ where at least one inclusion is proper) iff (E, F) is not strongly standard.

PROOF. (i) Since the graph norm of $e + f$ w.r.t. $s_{E,F}$ is $(\|e + f\|^2 + \|s_{E,F}(e + f)\|^2)^{1/2} = \sqrt{2}(\|e\|^2 + \|f\|^2)^{1/2}$, the result follows.

(ii) Immediate by the closed graph theorem.

(iii) Suppose (E, F) is strongly standard and let (\tilde{E}, \tilde{F}) be a standard extension. For any $y \in \tilde{E}$, $y = e + f$, hence $f = y - e \in F \cap \tilde{E} \subset \tilde{F} \cap \tilde{E}$, which implies $f = 0$, i.e. $y \in E$, hence $\tilde{E} = E$. \square

LEMMA 3.3. Let $N \leq_{\mathbb{R}} \mathcal{H}$ be a standard subspace. Then

(i) There exists a standard subspace M such that $N \subset M$ if and only if (N, iN) is not strongly standard.

(ii) If $y \in (N + iN)^c$, then $M = \{x + \lambda y : x \in N, \lambda \in \mathbb{R}\}$ is standard (but not strongly standard).

PROOF. If (N, iN) is strongly standard and $M \supset N$, then (M, iM) cannot be standard by Lemma 3.2, (iii).

Now let (N, iN) be not strongly standard, $y \in (N + iN)^c$ and $M = \{x + \lambda y : x \in N, \lambda \in \mathbb{R}\}$. If $z \in M \cap iM$ then $z = n_1 + \lambda_1 y = in_2 + i\lambda_2 y$ for some $n_1, n_2 \in N$, $\lambda_1, \lambda_2 \in \mathbb{R}$. However λ_1 and λ_2 should be zero, otherwise $y = (i\lambda_2 - \lambda_1)^{-1}(n_1 - in_2) \in N + iN$, hence $z \in N \cap iN = \{0\}$. Finally, if M is standard then (N, iN) is not strongly standard hence s_N is unbounded. Since $s_N \subset s_M$, s_M is unbounded, hence M is not strongly standard. \square

LEMMA 3.4. Let N be a factor subspace of \mathcal{H} . Then

(i) There exists a subspace M such that $N \subset M$ is irreducible an irreducible if and only if (N, N') is not strongly standard.

(ii) If $y \in (N + N')^c$, and $M = \{x + \lambda y : x \in N, \lambda \in \mathbb{R}\}$, then $N' \cap M = \{0\}$ (and (M, M') is not strongly standard).

PROOF. If (N, N') is strongly standard and $M \supset N$, then (M, N') cannot be standard by Lemma 3.2, (iii).

Now let (N, N') be not strongly standard, $y \in (N + N')^c$ and $M = \{x + \lambda y : x \in N, \lambda \in \mathbb{R}\}$. If $z \in M \cap N'$ then $z = n + \lambda y = n'$ for some $n \in N$, $n' \in N'$, $\lambda \in \mathbb{R}$. However λ should be zero, otherwise $y = \lambda^{-1}(n' - n) \in N + N'$, hence $z \in N \cap N' = \{0\}$. Finally, observe that (M, M') is properly extended by (M, N') , hence $s(M, M')$ cannot be bounded, which means that (M, M') is not strongly standard. \square

PROPOSITION 3.5. Let N be a standard subspace of \mathcal{H} . Then

(i) There exists a standard subspace $M \supset N$ if and only if $0 \in \sigma(\delta_N)$.

(ii) Suppose N is a factor. There exists a subspace M such that $N \subset M$ is an irreducible inclusion if and only if $1 \in \sigma(\delta_N)$.

(iii) Suppose N is a factor. There exists a standard subspace M such that $N \subset M$ is an irreducible inclusion if and only if $\{0, 1\} \subset \sigma(\delta_N)$.

(iv) If $N \subset M$ is an irreducible inclusion of standard spaces, then N and M are type III_1 factors.

PROOF. (i) By Lemmas 3.2, (ii), 3.3, (i), there exists a standard subspace $M \supset N$ iff s_N , hence $\delta_N^{1/2}$, is unbounded. Since $j_N \delta_N^{1/2} j_N = \delta_N^{-1/2}$, this is equivalent to $0 \in \sigma(\delta_N^{1/2})$, which in turn corresponds to $0 \in \sigma(\delta_N)$ by spectral mapping.

(ii) Let Θ be the selfadjoint operator defined by $\cos \Theta = \frac{|\delta - I|}{\delta + I}$, $\delta = \delta_N$, and let $s_{N,N'} = u d^{1/2}$ be the polar decomposition of $s_{N,N'}$. By $s_{N,N'}^2 \subset I$ one obtains $u^2 = 1$ and $u d u = d^{-1}$, and by [7], Proposition 3.2, one gets $\sin \Theta = \frac{|d - I|}{d + I}$. This implies that there exists a subspace M such that $N \subset M$ is irreducible $\iff s_{N,N'}$ is unbounded, $\iff 0 \in \sigma(d)$, $\iff \pi/2 \in \sigma(\Theta)$, $\iff 1 \in \sigma(\delta)$.

(iii) By the relations among δ , d and Θ one gets that $x \in \mathcal{D}(\delta^{1/2})^c \iff \chi_{[0,\varepsilon]}(\Theta)x \in \mathcal{D}(\delta^{1/2})^c$ and $y \in \mathcal{D}(d^{1/2})^c \iff \chi_{[\pi/2-\varepsilon,\pi/2]}(\Theta)y \in \mathcal{D}(d^{1/2})^c$, therefore if $x \in \mathcal{D}(\delta^{1/2})^c$ and $y \in \mathcal{D}(d^{1/2})^c$, then $z = \chi_{[0,\pi/4]}(\Theta)x + \chi_{[\pi/4,\pi/2]}(\Theta)y \in (\mathcal{D}(\delta^{1/2}) \cup \mathcal{D}(d^{1/2}))^c$, hence $M = \{x + \lambda z : x \in N, \lambda \in \mathbb{R}\}$ has the required properties by Lemmas 3.3, 3.4.

(iv) As shown before, N is a factor if and only if $1 \in \sigma(\delta_N)$. By [7], Proposition 4.5, this implies that N is an injective type III₁ factor. Since $M \subset j_M N' j_M$ is an irreducible inclusion, the same reasoning applies to M . \square

REMARK 3.6. If the inclusion of standard spaces $M_0 \subset M_1$ is irreducible, all the algebras of the Jones tower (and tunnel) are standard type III₁ factors.

We have proved that the set of irreducible inclusions of standard factors in \mathcal{H} is not empty. More precisely we have shown how to construct irreducible inclusions of standard factors with codimension 1. Iterating the process and taking direct sums one can construct irreducible inclusions of standard factors with any codimension. In the following we study the structure of such inclusions.

LEMMA 3.7. *If M is standard in \mathcal{H} , $(M, \text{Ker}(j + I))$ is a strongly standard pair, where j is the modular conjugation of M .*

PROOF. This proof closely follows the proof of Proposition 3.2 in [7]. The spaces $M, \text{Ker}(j + I)$ are both reduced by the spectral projections relative to $|\log \delta|$, where δ is the modular operator relative to M , hence it is sufficient to study the problem on the fiber of the representation of \mathcal{H} as direct integral w.r.t. $|\log \delta|$. Then, as in [7], Prop. 3.2, it is sufficient to study the case in which K is a standard subspace of \mathbb{C}^2 , hence K can be seen as generated by the vectors

$$y^+ = \begin{pmatrix} \cos \frac{\vartheta}{2} \\ \sin \frac{\vartheta}{2} \end{pmatrix} \quad y^- = \begin{pmatrix} i \cos \frac{\vartheta}{2} \\ -i \sin \frac{\vartheta}{2} \end{pmatrix}$$

and j is represented by the matrix

$$\begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$$

where C is the complex conjugation. Therefore

$$\text{Ker}(j + I) = \left\{ \begin{pmatrix} a \\ -\overline{a} \end{pmatrix}, \quad a \in \mathbb{R} \right\}$$

and it is easy to see that

$$|\text{Re}(h, k)| \leq \frac{\sqrt{2}}{2}, \quad h \in M, \quad k \in \text{Ker}(j + I), \quad \|h\| = \|k\| = 1,$$

which implies

$$\begin{aligned} \frac{\|h+k\|^2}{\|h\|^2 + \|k\|^2} &= \frac{\|h\|^2 + \|k\|^2 + 2\operatorname{Re}(h, k)}{\|h\|^2 + \|k\|^2} \\ &\geq \frac{\|h\|^2 + \|k\|^2 - \sqrt{2}\|h\|\|k\|}{\|h\|^2 + \|k\|^2} \\ &\geq \frac{\sqrt{2}-1}{\sqrt{2}}, \end{aligned}$$

namely the graph norm of $s_{M, \operatorname{Ker}(j+I)}$ is equivalent to the Hilbert norm, and the strongly standard property follows by Lemma 3.2, (ii). \square

If $M_0 \subset M_1$ are standard subspaces and j_0, j_1 are the modular conjugations, the tower and the tunnel of standard subspaces are inductively defined by the equations:

$$M_{k+1} = j_k M'_{k-1}, \quad k \in \mathbb{Z}.$$

We shall use the following notations:

$$(3.1) \quad A_{k,l} = M'_k \cap M_l \quad k \leq l \in \mathbb{Z}$$

$$(3.2) \quad B_k = M_{k+1} \cap \operatorname{Ker}(j_k + I) \quad k \in \mathbb{Z}$$

Now we can prove some structural results about inclusions of standard subspaces.

PROPOSITION 3.8. *Let $M_0 \subset M_1$ be an inclusion of standard subspaces. Then the following holds:*

$$(i) \quad M_{k+p} = M_k + \sum_{j=0}^{p-1} B_{k+j} \quad k \in \mathbb{Z}, p \geq 0$$

$$(ii) \quad A_{k-1, k+p} = A_{k-1, k} + \sum_{j=0}^{p-1} B_{k+j} \quad k \in \mathbb{Z}, p \geq 0$$

$$(iii) \quad B_k \text{ is commutative} \quad k \in \mathbb{Z}$$

$$(iv) \quad \sum_{j=1}^{2p} B_{k+j} \text{ is a factor} \quad k \in \mathbb{Z}, p > 0$$

$$(v) \quad \mathcal{Z} \left(\sum_{j=1}^{2p+1} B_{k+j} \right) = \left\{ \left(I + \sum_{j=1}^p (-1)^j j_{k+2j} \right) x : x \in B_{k+1} \right\} \quad k \in \mathbb{Z}, p > 0,$$

where $\mathcal{Z}(P) = P \cap P'$. In particular, if the inclusion $M_0 \subset M_1$ is irreducible,

$$(vi) \quad A_{k-1, k+1} = B_k \text{ is abelian } \forall k \in \mathbb{Z}$$

$$(vii) \quad A_{k-1, k+2} = B_k + B_{k+1} \text{ is a factor } \forall k \in \mathbb{Z}$$

PROOF. By Lemma 3.7 and eq. (3.2), any vector in M_{k+p} can be (uniquely) decomposed in sum of a vector in M_{k+p-1} and a vector in B_{k+j-1} . Iterating this argument we get (i).

Now we note that, by the global j_k -invariance of B_k ,

$$B_k \subseteq M_{k+1} \cap j_k M_{k+1} = M_{k+1} \cap M'_{k-1} = A_{k-1, k+1}.$$

In particular, B_{k+j} commutes with M_{k-1} for each $j \geq 0$. Therefore, applying the decomposition (i) to a vector in $A_{k-1, k+p}$, we get (ii).

Let us take $x, y \in B_k$. Since B_k is j_k -antiinvariant,

$$Im(x, y) = Im(x, -j_k y) = Im(y, -j_k x) = Im(y, x) = -Im(x, y),$$

which implies (iii).

Now observe that

$$\begin{aligned} B_{k+1} \cap B'_k &= (B_{k+1} \cap M'_k) \cap B'_k = B_{k+1} \cap (M_k \vee B_k)' = \\ &= B_{k+1} \cap M'_{k+1} = (M_{k+1} \cap M'_{k+1}) \cap Ker(j_{k+1} + I) \end{aligned}$$

Since the elements in the center of M_{k+1} are j_{k+1} invariant (see Remark 1.7, [7]), it turns out that $B_{k+1} \cap B'_k = 0$.

Then notice that $x \in \mathcal{Z}(\sum_{j=1}^p B_{k+j})$ iff $x \in \sum_{j=1}^p B_{k+j}$ and

$$(3.3) \quad Im(x, y) = 0 \quad \forall y \in B_{k+j}, \quad 1 \leq j \leq p.$$

Since any $x \in \sum_{j=1}^p B_{k+j}$ may be uniquely written as $x = \sum_{j=1}^p b_{k+j}$, $b_{k+j} \in B_{k+j}$, and B_m commutes with B_n if $|m - n| \geq 2$, eq. (3.3) for $j = 1$ implies $b_{k+2} = 0$. Then eq. 3.3 for $j = 3$ implies $b_{k+4} = 0$ and, iterating, $b_{k+2j} = 0$ for any $1 \leq j \leq p/2$. Analogously we show that b_{k+p-1} vanishes, and therefore $b_{k+p-2j-1} = 0$ for $j \leq p/2 - 1$. If p is even this shows that $x = 0$, i.e. (iv).

When $p = 2m + 1$ is odd, we proved that $x = \sum_{j=0}^m b_{k+2j+1}$. Now, taking $y \in B_{k+2j}$, $1 \leq j \leq m$, and making use of the j -anti-invariance of B , we have

$$\begin{aligned} 0 &= Im(x, y) = Im(b_{k+2j-1} + b_{k+2j+1}, y) \\ &= Im(b_{k+2j-1}, y) - Im(b_{k+2j+1}, j_{k+2j} y) \\ &= Im(b_{k+2j-1}, y) + Im(j_{k+2j} b_{k+2j+1}, y) \\ &= Im(b_{k+2j-1} + j_{k+2j} b_{k+2j+1}, y), \end{aligned}$$

hence $b_{k+2j-1} + j_{k+2j} b_{k+2j+1} = 0$, and this implies (v).

Finally we observe that the inclusion $M_0 \subset M_1$ is irreducible iff $A_{0,1}$, and therefore $A_{k,k+1}$, $\forall k \in \mathbb{Z}$, are trivial. Then (vi) follows from (ii) and (iii) and (vii) follows from (ii) and (iv). \square

4. Inclusions of second quantization algebras

We first recall that an inclusion of von Neumann algebras $\mathcal{M}_0 \subset \mathcal{M}_1$ is said of depth 2 if the von Neumann algebra $\mathcal{M}'_0 \cap \mathcal{M}_3$ is a factor.

THEOREM 4.1. *Let $M_0 \subset M_1$ be an inclusion of standard subspaces.*

(i) *If the codimension of M_0 in M_1 is finite and equal to n*

$$\mathcal{R}(M_1) = \mathcal{R}(M_0) \times_{\alpha} \mathbb{R}^n$$

where \mathbb{R}^n is identified with B_0 and $\alpha_h(\cdot) = \text{ad } W(h)$, $h \in B_0$.

(ii) *If $M_0 \subset M_1$ is irreducible then $\mathcal{R}(M_0) \subset \mathcal{R}(M_1)$ is of depth two.*

PROOF. (i) First we note that, by eq. 2.2,

$$(4.1) \quad \text{ad } W(h)(W(k)) = e^{-iIm(h,k)} W(h+k)$$

i.e. each closed commutative subspace of \mathcal{H} considered as an additive group acts on any second quantization algebra via $\text{ad } W(\cdot)$.

Then we only have to check that the hypotheses of Landstad Theorem [12] are satisfied. By Proposition 2.1 and Proposition 3.8 (i) we get

$$\mathcal{R}(M_1) = \mathcal{R}(M_0) \vee \{W(h) : h \in B_0\}''$$

i.e. $\mathcal{R}(M_1)$ is generated by $\mathcal{R}(M_0)$ and a unitary representation of B_0 which is strongly continuous by the finite-dimensionality of B_0 .

Now we claim that the action

$$\beta : h \in B_1 \rightarrow \text{ad } W(h) \in \text{Aut}(\mathcal{R}(M_1))$$

is dual to the action α of B_0 on M_0 .

On the one hand

$$\mathcal{R}(M_1)^\beta = \mathcal{R}(M_1) \cap \{W(h) : h \in B_1\}' = \mathcal{R}(M_1 \cap B_1') = \mathcal{R}(M_0)$$

where the last equality follows from Proposition 3.8 (i), (iv). On the other hand, by eq. (4.1), B_1 acts as the dual group on $\mathcal{R}(B_0)$ iff the pairing

$$\text{Im}(\cdot, \cdot) : B_0 \times B_1 \rightarrow \mathbb{R}$$

is a duality between real Hilbert spaces. The factoriality of $B_0 \vee B_1$ implies that the pairing is non degenerate, the finite-dimensionality of B_0 and B_1 implies that it is continuous, and the thesis follows.

(ii) follows immediately by Proposition 3.8 □

4.1. Non regular inclusions.

THEOREM 4.2. *There exists an irreducible inclusion $M_0 \subset M_1$ of standard subspaces with infinite codimension for which the third relative commutant is of type III.*

PROOF. Let $M_0^n \subset M_1^n$ be a sequence of irreducible inclusions of standard subspaces such that the codimension of M_0^n in M_1^n is one, and let M_0 , resp. M_1 the direct sum of the M_0^n , resp. M_1^n in $\oplus_{n=1}^\infty \mathcal{H}$.

As it is well known, direct sums of complex orthogonal real subspaces give rise to tensor products at the second quantization level, therefore the decomposition $A_{0,3} = \oplus_{n=1}^\infty A_{0,3}^n$ gives rise to an ITPFI decomposition. Since the codimension of M_0^n in M_1^n is one, $A_{0,3}^n$ is two-dimensional and the angle of $A_{0,3}^n$ with $iA_{0,3}^n$ is a number θ_n .

Therefore, by [2] and [7] Proposition 2.7, the type of $A_{0,3}$ depends only on the sequence θ_n . Choosing isomorphic inclusions $M_0^n \subset M_1^n$ one gets a constant sequence θ_n hence the factor has type III_λ , with $\lambda = \tan^2 \theta/2$. □

Let us recall that an inclusion $\mathcal{M}_0 \subset \mathcal{M}_1$ is called *regular* if the space of intertwiners $T : yJ_1xJ_1T = TyJ_2xJ_2$ for any $x \in \mathcal{M}_1$, $y \in \mathcal{M}_0$ is non zero. It was shown in [5] that for any depth two regular inclusion the third relative commutant is of type I, and that for any crossed-product inclusion regularity holds. This shows that the inclusions of Theorem 4.2 are not regular, hence cannot come from a crossed-product with a locally compact group. Indeed they constitute an example of a kind of crossed-product with the group given by an infinite dimensional Hilbert space with the additive structure.

We conclude this paper observing that, for the described inclusions, the only visible invariant is the type of the third relative commutant. Since these inclusions are, in a sense, topologically trivial, it seems reasonable to conjecture that the third relative comutant is actually a complete invariant.

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D. GUIDO, DIP. DI MATEMATICA, UNIVERSITÀ DELLA BASILICATA, I–10085 POTENZA, ITALY.
E-mail address: `guido@unibas.it`